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Some Results on Causal Modalities in General Spacetimes

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Inria, Laboratoire Méthodes Formelles

Minkowski Spacetime + Modal Logic

$$x \preccurlyeq y \quad \text{iff} \quad \sum_{i=1}^{n-1}(y_i - x_i)^2 \leq (y_0 - x_0)^2 \text{ and } x_0 \leq y_0$$

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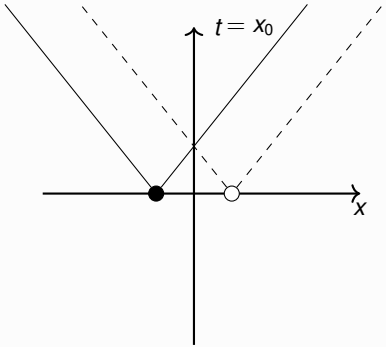

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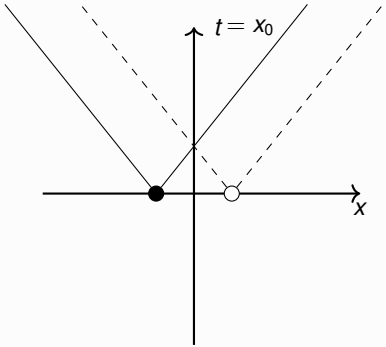

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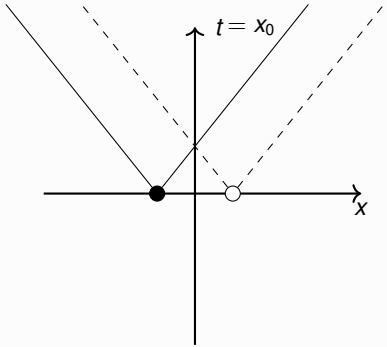

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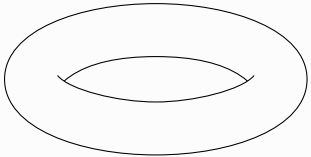
Shapirovsky and Shehtman, “Chronological Future Modality in Minkowski Spacetime”, 2002

General Spacetimes



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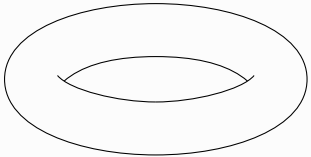

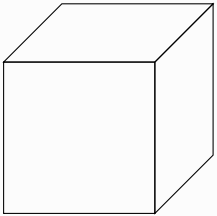

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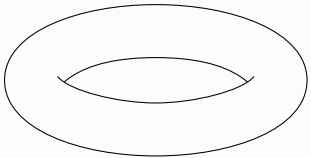

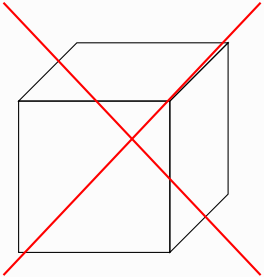

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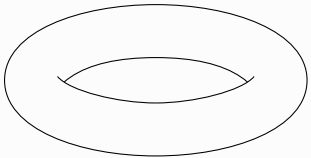

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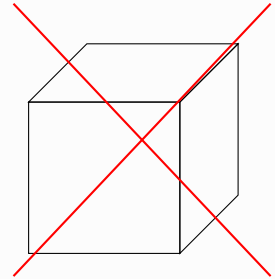

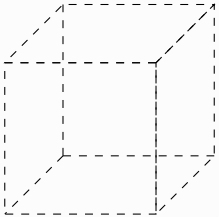

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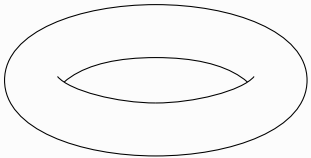

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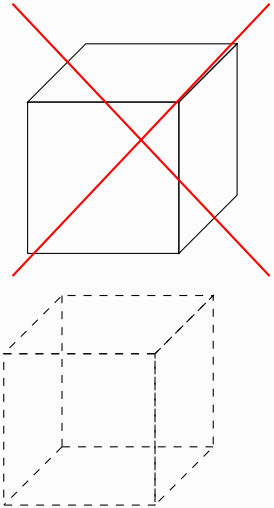

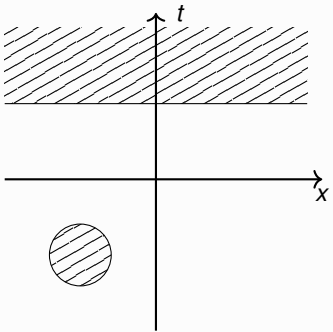

Main Research Questions

- What properties are inherent in all spacetimes?
- How do the logics change as the physical/causal properties of spacetime changes?

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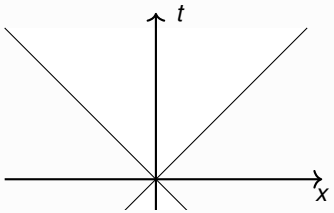

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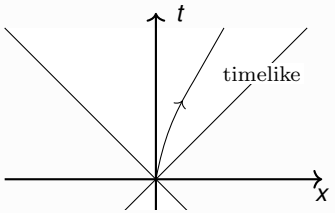

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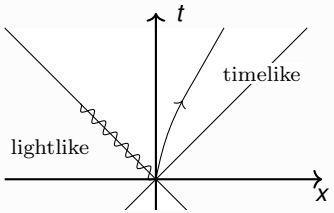

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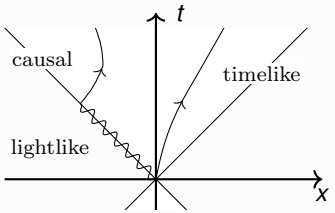

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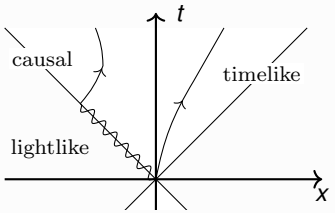

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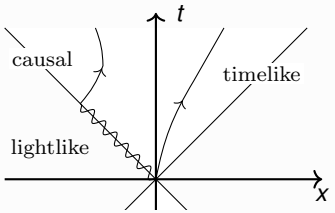

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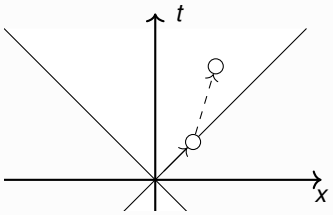

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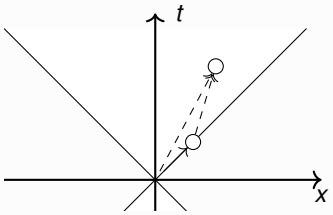

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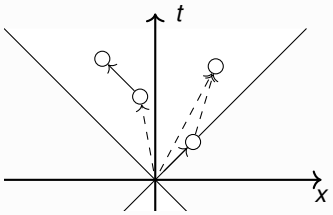

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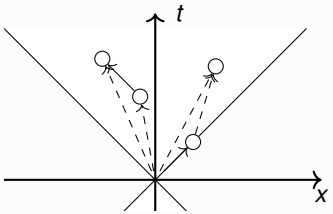
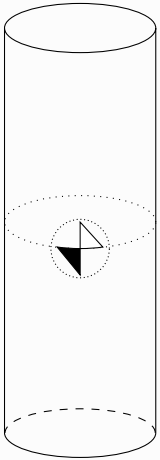


Local Minkowski Space in General Spacetimes



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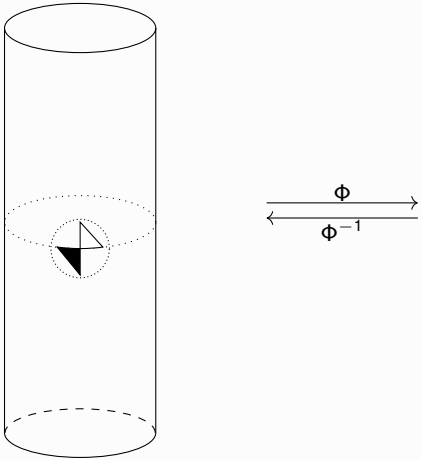

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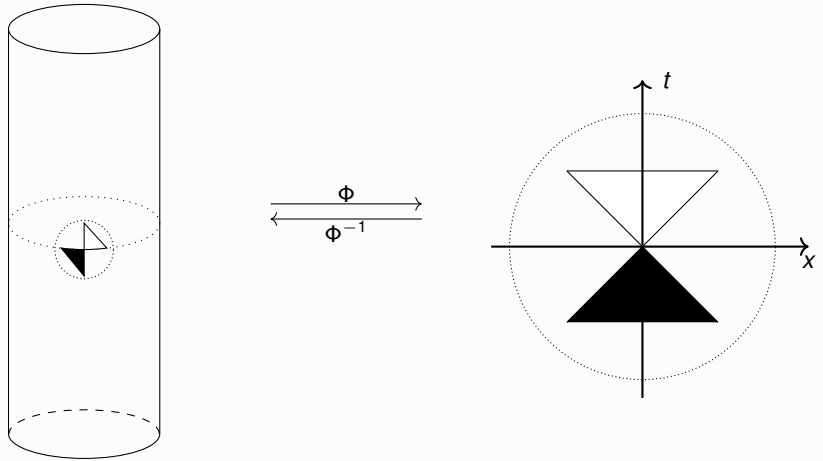

General Spacetimes + Modal Logic (Trivial Results)

Corollary (4.1)

Let M be a spacetime. Then, we have that

(i) **S4** $\subseteq \mathcal{L}(\langle M, \ll\!\!\ll \rangle), \mathcal{L}(\langle M, \preccurlyeq \rangle)$;

(ii) **OI** $\subseteq \mathcal{L}(\langle M, \ll \rangle)$;

(iii) **D4** $+$ *ad* $\subseteq \mathcal{L}(\langle M, \alpha \rangle)$.

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$$a\alpha f := \Diamond(\Diamond(p_1 \wedge \neg p_2 \wedge \Box\neg p_2) \wedge \Diamond(p_2 \wedge \neg p_1 \wedge \Box\neg p_1)) \wedge \Diamond q$$
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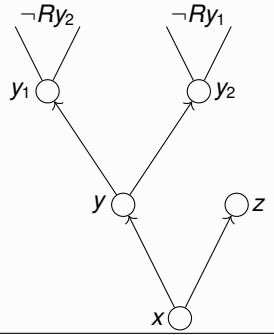
This formula is Sahlqvist (so canonical) and corresponds to the depicted FO-property:



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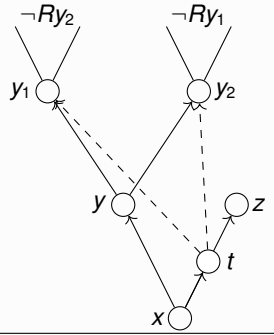

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$$\langle \mathbb{R}^n, \alpha \rangle \models a\alpha f$$

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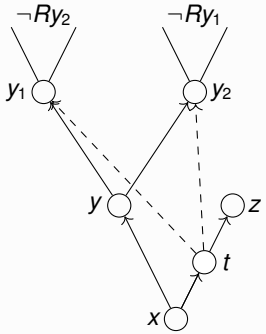

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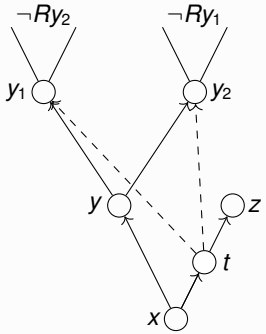

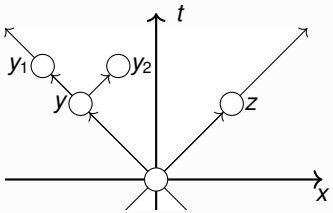

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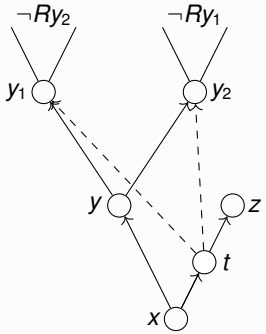

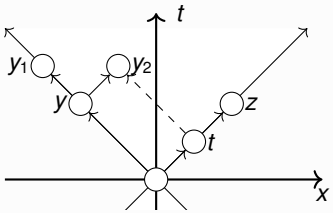

After Formula in General Spacetime

Want to show for a general spacetime, M with α:

$$\langle M, \alpha \rangle \models a \alpha f$$

– Have $x \; \alpha \; y \; \alpha \; y_1, y_2$, $x \; \alpha \; z$; want t such that $x \; \alpha \; t \; \alpha \; y_i, z$

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- Main observation: if $x \ll y_1$, y_2, or z, can find t easily
- In case where $x \ll y \; \alpha \; y_1, y_2$ can use Push-up Rule to get $x \ll y_1$

Push-Up Rule

$$x \ll y \preccurlyeq z \implies x \ll z$$

$$x \preccurlyeq y \ll z \implies x \ll z$$

After Formula in General Spacetime

Want to show for a general spacetime, M with α:

$$\langle M, \alpha \rangle \models a \alpha f$$

- Have $x \; \alpha \; y \; \alpha \; y_1, y_2$, $x \; \alpha \; z$; want t such that $x \; \alpha \; t \; \alpha \; y_i, z$
- Main observation: if $x \ll y_1$, y_2, or z, can find t easily
- In case where $x \ll y \; \alpha \; y_1, y_2$ can use Push-up Rule to get $x \ll y_1$
- Main case to consider: $x \to y \to y_1, y_2$, $x \to z$

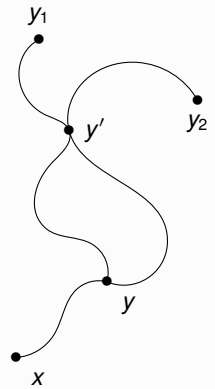
Push-Up Rule

$$x \ll y \preccurlyeq z \implies x \ll z$$

$$x \preccurlyeq y \ll z \implies x \ll z$$

After Formula in General Spacetime

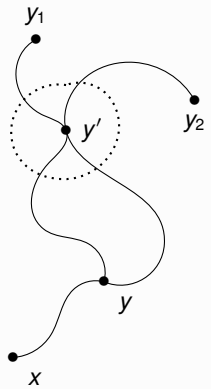
– Consider $x \to y \to y_1, y_2$,
$x \to z$



Global Spacetime

After Formula in General Spacetime

- Consider $x \to y \to y_1, y_2$, $x \to z$
- Take neighbourhood around last common point (y')



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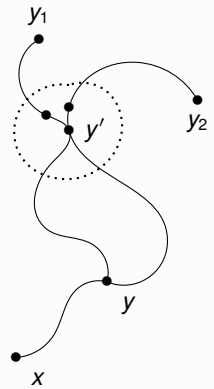

Global Spacetime

After Formula in General Spacetime

- Consider $x \to y \to y_1, y_2$, $x \to z$
- Take neighbourhood around last common point (y')
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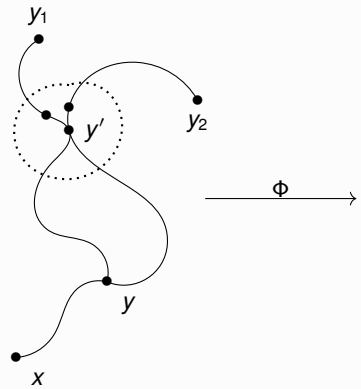

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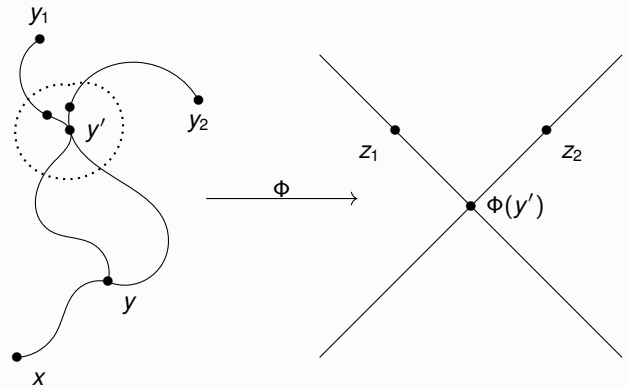

Global Spacetime

Local Minkowskian

After Formula in General Spacetime

- Consider $x \to y \to y_1, y_2$, $x \to z$
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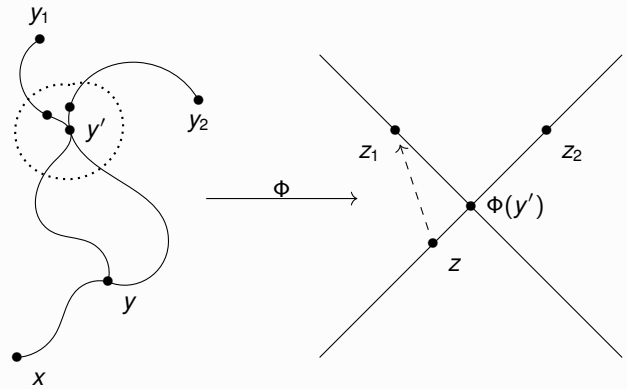
- Use lemma for Minkowski spacetime to get z



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- Consider $x \to y \to y_1, y_2$, $x \to z$
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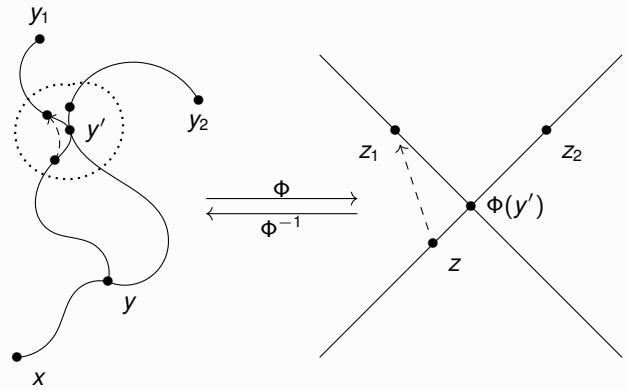

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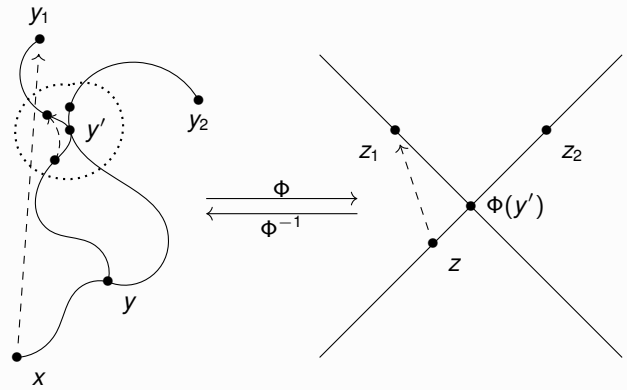

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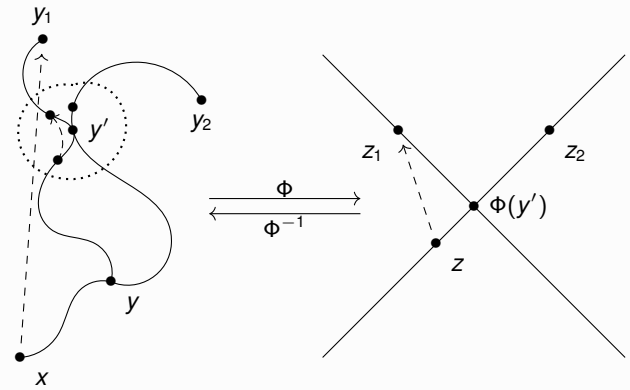
Obtain $\langle M, \alpha \rangle \models a\alpha f$



Global Spacetime

Local Minkowskian

2-D After Formula

$$a\alpha f := \Diamond(\Diamond(p_1 \wedge \neg p_2 \wedge \Box\neg p_2) \wedge \Diamond(p_2 \wedge \neg p_1 \wedge \Box\neg p_1)) \wedge \Diamond q$$
$$\rightarrow \Diamond(\Diamond p_1 \wedge \Diamond q) \vee \Diamond(\Diamond p_2 \wedge \Diamond q),$$

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Is also Sahlqvist and corresponds to FO-property:

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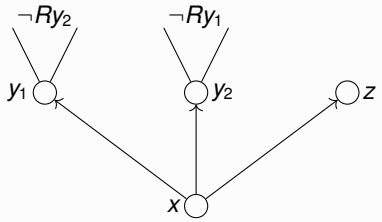

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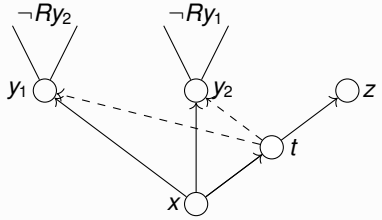
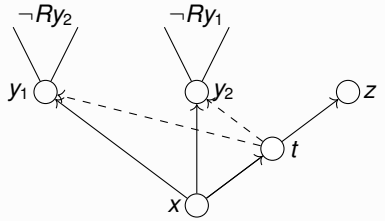


2-D After Formula for Minkowski Spacetime

$$\langle \mathbb{R}^2, \alpha \rangle \models a\alpha_2 f$$



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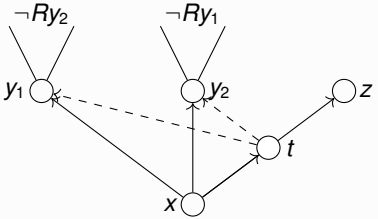

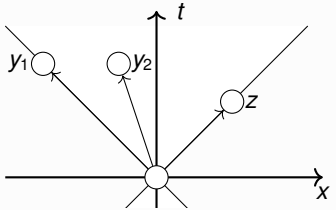

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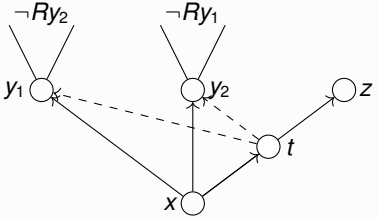

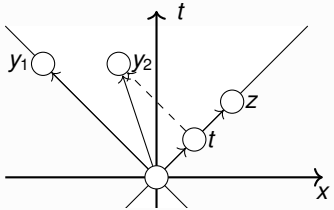

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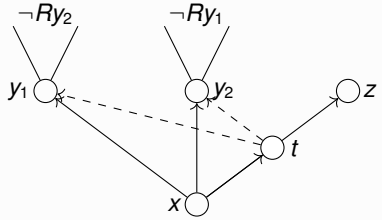

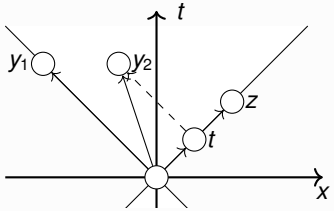

Shapirovsky and Shehtman, “Modal Logics of Regions and Minkowski Spacetime”, 2005

2-D After Formula for General Spacetimes

For a general 2-dimensional spacetime M:

$$\langle M, \alpha \rangle \models a\alpha_2 f$$

Have $x \ \alpha \ y_1, y_2, z$ and $\neg y_i \ \alpha \ y_j$; want t such that $x \ \alpha \ t \ \alpha \ y_i, z$

Proof is essentially a combination of:

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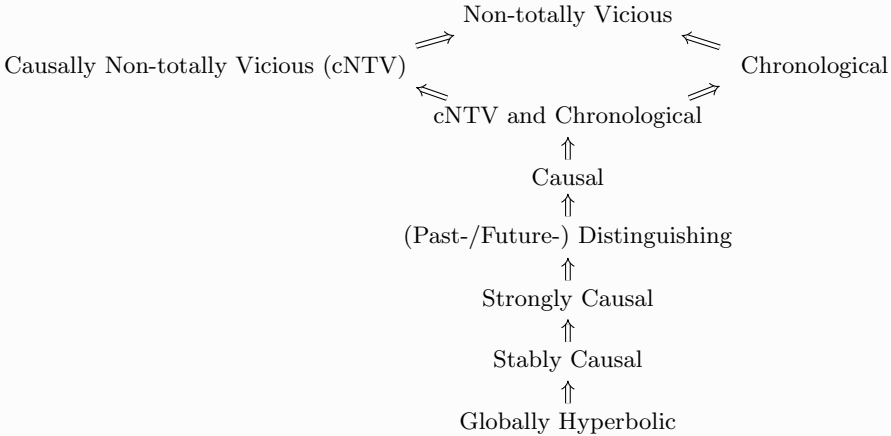
For $n \geq 3$: $\langle \mathbb{R}^n, \alpha \rangle \not\models a\alpha_2 f$; for arbitrary n-dimensional spacetime: probably $\langle M, \alpha \rangle \not\models a\alpha_2 f$

The Causal Ladder



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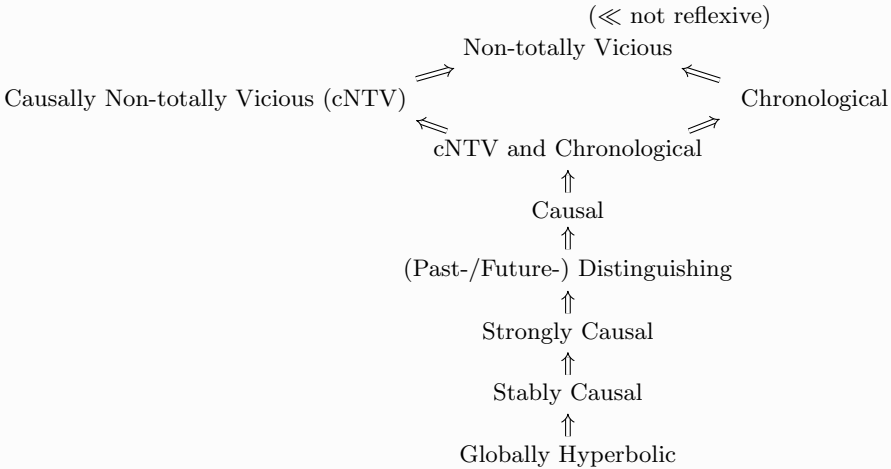

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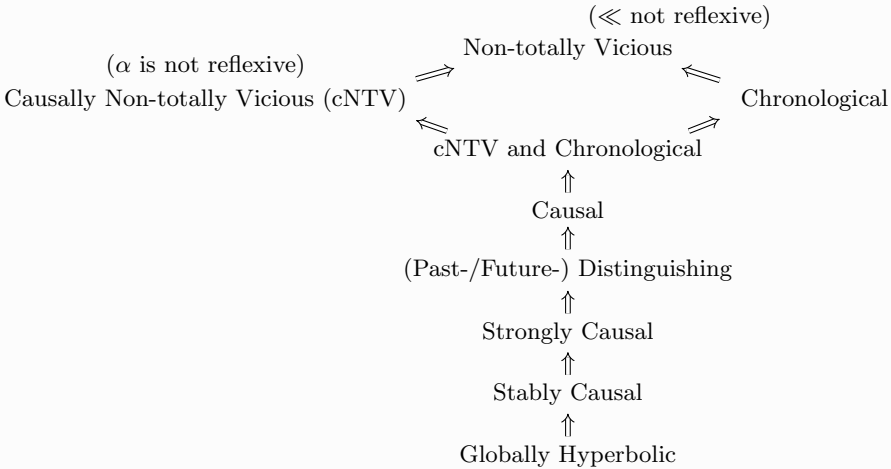

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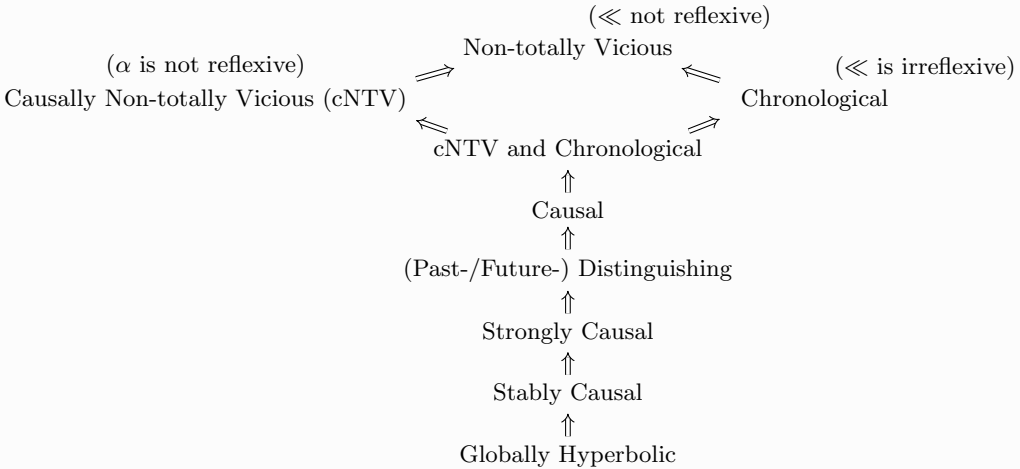

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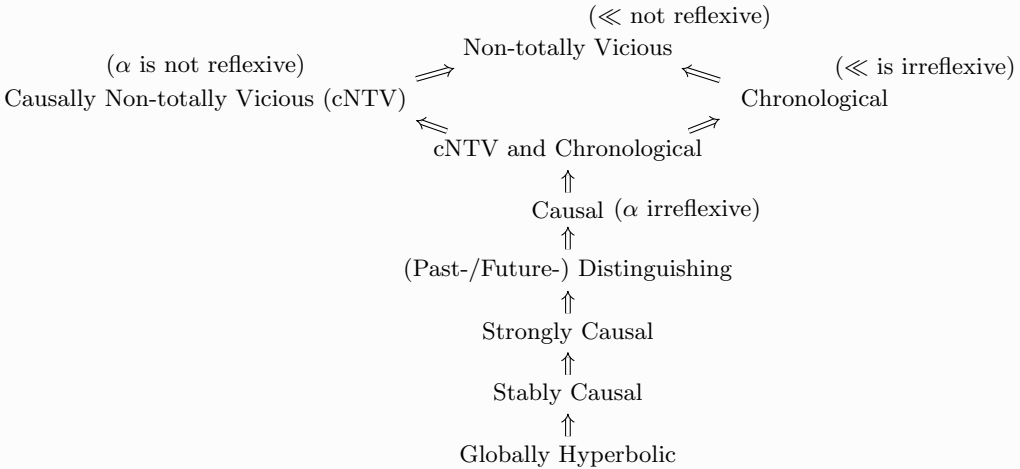

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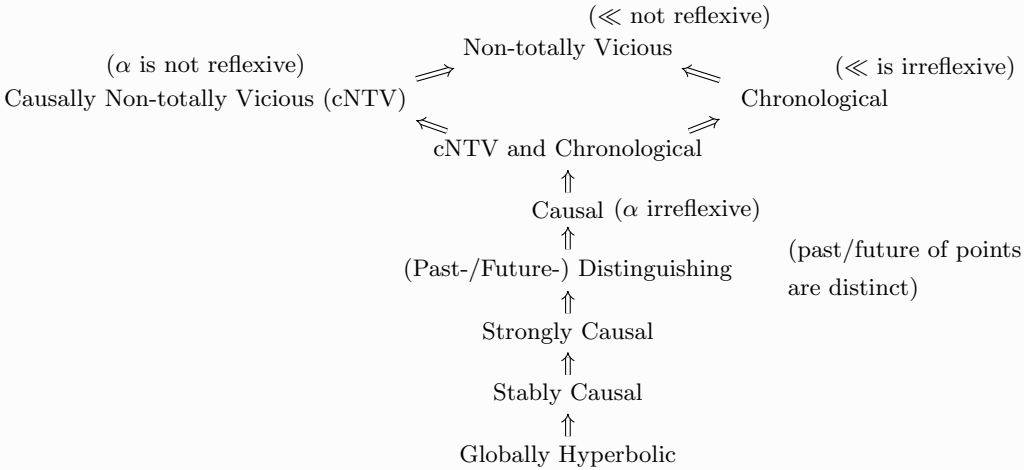

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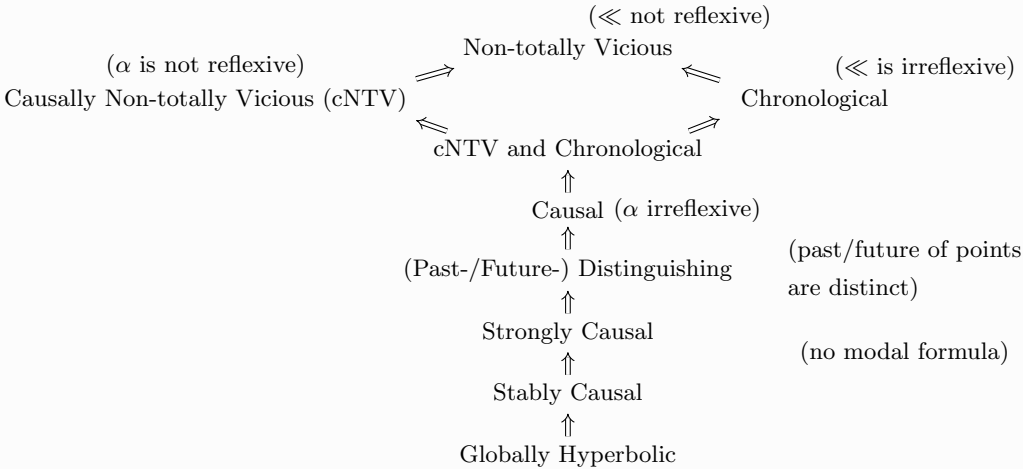

Modal Logics on the Causal Ladder

D4da $:=$ **D4** $+$ *ad* $(\Diamond p \to \Diamond\Diamond p) +$ *a*α*f*	Spacetime Modal Logic Containment ($L = \mathcal{L}(\langle M, \lhd \rangle)$)		
Spacetime Type (M)	$\ll$	α	$\not\ll$ / $\not\preccurlyeq$
No class			
Non-Totally Vicious			
Chronological			
Causally Non-Totally Vicious			
cNTV and Chronological			
Casual			
Distinguishing			
$\vdots$	$\vdots$		
Globally hyperbolic			

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D4da $:=$ **D4** $+ ad\ (\Diamond p \to \Diamond\Diamond p) + a\alpha f$	Spacetime Modal Logic Containment ($L = \mathcal{L}(\langle M, \lhd \rangle)$)		
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No class	**S4** $\subseteq L$	**S4** $\subseteq L$	**S4** $\subseteq L$
Non-Totally Vicious			
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Chronological	$\mathbf{OI} + (\text{Gabb}) \subseteq L$		
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Distinguishing			
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Globally hyperbolic			

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Distinguishing			
$\vdots$	$\vdots$		
Globally hyperbolic	**OI** $+$ (Gabb) $\subseteq L$	**D4da** $+$ (Gabb) $\subseteq L$	**S4** $\subseteq L$

Conclusion

– Showed that key property enjoyed by Minkowski spacetime ($a\alpha f$) are also enjoyed by general spacetimes

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Conclusion

- Showed that key property enjoyed by Minkowski spacetime ($a\alpha f$) are also enjoyed by general spacetimes
- Showed that unique property enjoyed by 2-D Minkowski spacetime ($a\alpha_2 f$) are also enjoyed by general 2-D spacetimes
- Investigated how modal logics of causal relations change on the causal ladder

Open Questions

Modal Logic of α in Minkowski space

$\mathcal{L}(\langle \mathbb{R}^2, \alpha \rangle) =$ **D4** $+ ad + a\alpha_2 f +$?

$\mathcal{L}(\langle \mathbb{R}^n, \alpha \rangle) =$?

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Completeness

For M non-totally vicious spacetime:

$$\mathcal{L}(\langle M, \preccurlyeq \rangle) \subseteq \textbf{S4.2}?$$

(similarly for $\ll$ with **OI.2**, α with ?.**2**)

Open Questions

Modal Logic of α in Minkowski space

$$\mathcal{L}(\langle \mathbb{R}^2, \alpha \rangle) = \mathbf{D4} + ad + a\alpha_2 f + ?$$

$$\mathcal{L}(\langle \mathbb{R}^n, \alpha \rangle) = ?$$

Modal link with Push-up Rule

$$x \ll y \preccurlyeq z \implies x \ll z$$

$$x \preccurlyeq y \ll z \implies x \ll z$$

Corresponds to formulae in modal logic?

Completeness

For M non-totally vicious spacetime:

$$\mathcal{L}(\langle M, \preccurlyeq \rangle) \subseteq \mathbf{S4.2}?$$

(similarly for $\ll$ with **OI.2**, α with ?.**2**)

Open Questions

Modal Logic of α in Minkowski space

$$\mathcal{L}(\langle \mathbb{R}^2, \alpha \rangle) = \mathbf{D4} + ad + a\alpha_2 f + ?$$

$$\mathcal{L}(\langle \mathbb{R}^n, \alpha \rangle) = ?$$

Modal link with Push-up Rule

$$x \ll y \prec z \implies x \ll z$$

$$x \prec y \ll z \implies x \ll z$$

Corresponds to formulae in modal logic?

Completeness

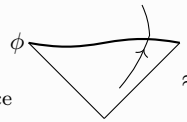
For M non-totally vicious spacetime:

$$\mathcal{L}(\langle M, \prec \rangle) \subseteq \mathbf{S4.2}?$$

(similarly for $\ll$ with **OI.2**, α with ?.**2**)

Extensions to more expressive logics (CTL, tense logic, ...)

Domain of Dependence



$\approx \mathbf{AF}\phi$

(3,2)-Density and $a\alpha_2 f$

$$ad_{3,2} := \Diamond p_1 \wedge \Diamond p_2 \wedge \Diamond p_3 \rightarrow \Diamond(\Diamond p_1 \wedge \Diamond p_2) \vee \Diamond(\Diamond p_1 \wedge \Diamond p_3) \vee \Diamond(\Diamond p_2 \wedge \Diamond p_3)$$

For normal modal logic, Λ,

$$\Lambda + ad_{3,2} + a\alpha f \equiv \Lambda + ad + a\alpha_2 f$$

(may require Λ to be transitive.)